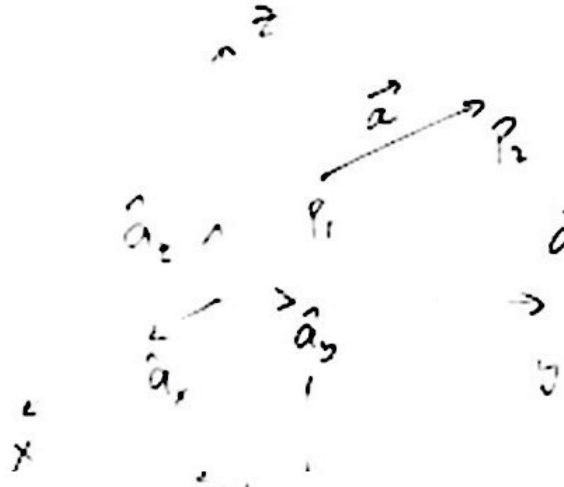
# Vectors:




$\vec{a}$ = direction + magnitude.

Unit vectors magnitude = 1

direction in the major axis.

$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

Direction unit vector

Magnitude

Then,

$$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \hat{a}_x(x_2 - x_1) + \hat{a}_y(y_2 - y_1) + \hat{a}_z(z_2 - z_1$$

Components of the vector $\vec{a}$

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Where

$\hat{a}_x, \hat{a}_y, \hat{a}_z$ are unit vectors.

$|\hat{a}_x|$ = Magnitude of $\hat{a}_x$

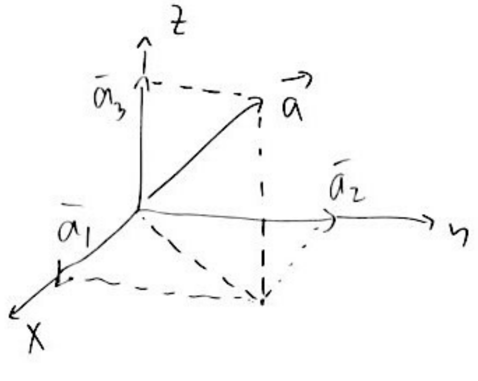
$= 1$

$|\hat{a}_y| = 1, \quad |\hat{a}_z| = 1$

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

Then

$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$



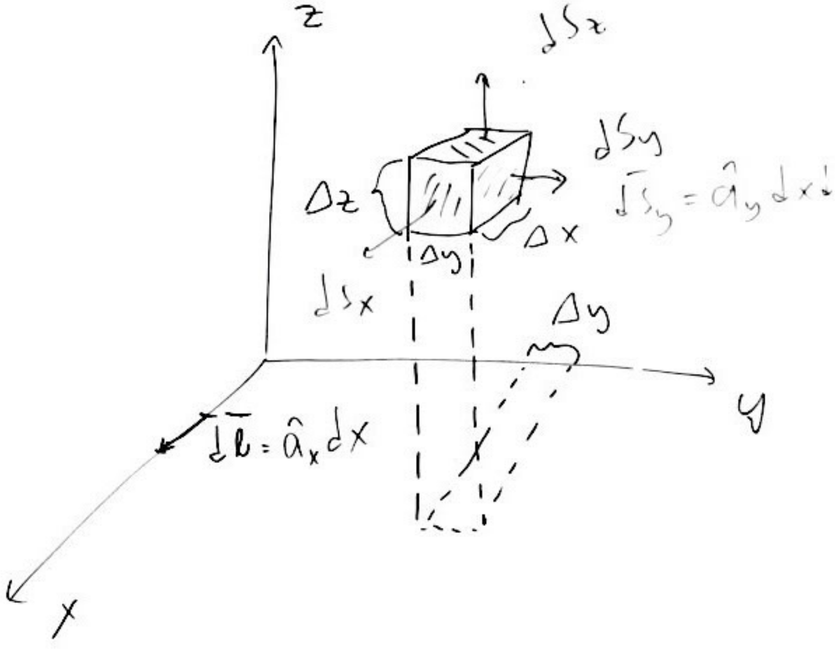
Differentials:

$\Delta x, \Delta y, \Delta z$

In the limit case

$\lim \Delta x = dx$

$\Delta x = x_1 - x_2 \to 0$

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## Coordinate Systems:

1) Rectangular (Cartesian) Coord. System.

Base axises are x, y, z.

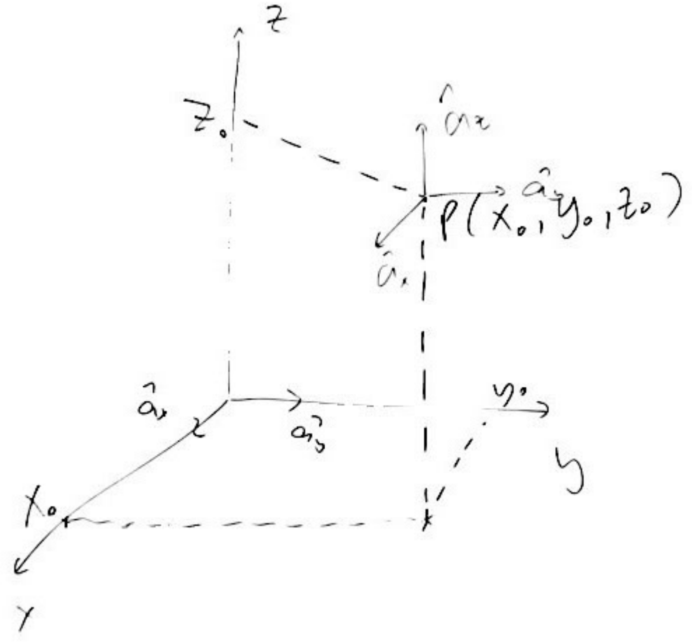
A point $P(x_0, y_0, z_0)$

Base vectors:

$\hat{a}_x, \hat{a}_y, \hat{a}_z$

$$\hat{a}_x \cdot \hat{a}_y = 0$$
$$\hat{a}_y \cdot \hat{a}_z = 0$$
$$\hat{a}_x \cdot \hat{a}_z = 0$$

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$
$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$
$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$



Differential lengths:

$dx, \ dy, \ dz$

Differential areas:

$dS_x = dy\,dz$

$dS_y = dx\,dz$

$dS_z = dx\,dy$

Vector Differentials:

$\bar{dl} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$ (Diff. length vector)

Vector Surface Differentials:

$\bar{dS}_x = \hat{a}_x dy\,dz$

$\bar{dS}_y = \hat{a}_y dx\,dz$

$\bar{dS}_z = \hat{a}_z dx\,dy$

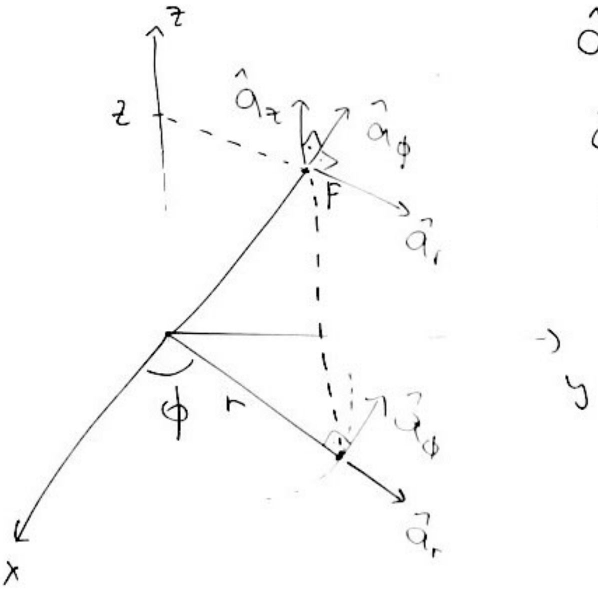
2-) Cylindrical Coord. System:

Base axis are $r, \phi, z$

$(\rho)$

$\hat{a}_r, \hat{a}_\phi, \hat{a}_z =$ Base vectors.

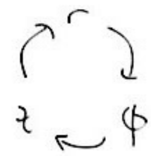
A point $P(r, \phi, z)$




$\hat{a}_r \cdot \hat{a}_\phi = 0$

$\hat{a}_r \cdot \hat{a}_z = 0$

$\hat{a}_\phi \cdot \hat{a}_z = 0$

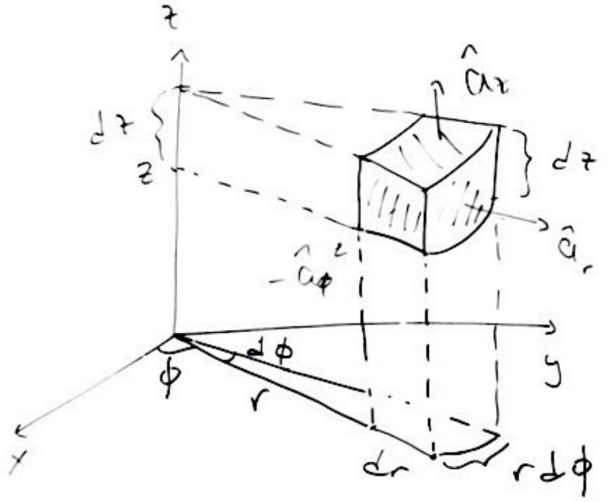



$\hat{a}_r \times \hat{a}_\phi = \hat{a}_z$ ;

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_r$

$\hat{a}_z \times \hat{a}_r = \hat{a}_\phi$

# Differentials:




## Diff. Lengths:

$dr,\ rd\phi,\ dz$

## Diff. Areas:

$dS_r = rd\phi\, dz$

$dS_\phi = dr\, dz$

$dS_z = r\, d\phi\, dr$

$= r\, dr\, d\phi$

## Vector Differentials:

$\bar{dl} = \hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz$

and the surface diff. vectors are.

$\bar{dS}_r = \hat{a}_r\, rd\phi dz\ ,\ \bar{dS}_\phi = \hat{a}_\phi dr dz\ ,\ \bar{dS}_z = r dr d\phi$

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Given a point $P(x, y, z)$

and $P(r, \phi, z)$ can be

related as

$$r^2 = x^2 + y^2, \quad \phi = \tan^{-1}\frac{y}{x}, \quad z = z$$

and

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z.$$

Given a vector $\bar{A} = \langle A_x, A_y, A_z \rangle$

and $\bar{A} = \langle A_r, A_\phi, A_z \rangle$ are

related as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

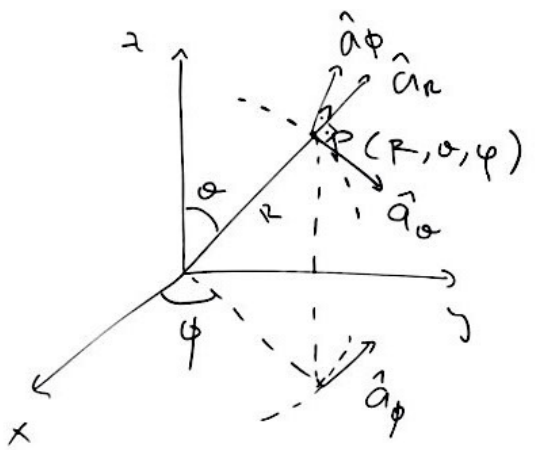
and

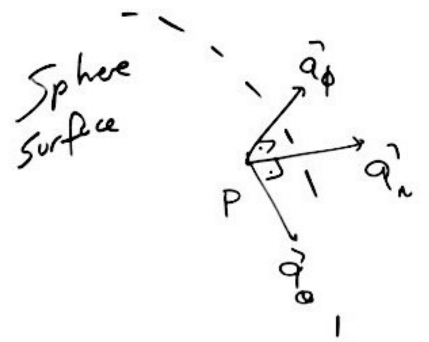
$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Base vectors are $\hat{a}_R, \hat{a}_\theta, \hat{a}_\phi$

Coordinate variables are : $R, \theta, \phi$

A point $P(R, \theta, \phi)$ is shown as

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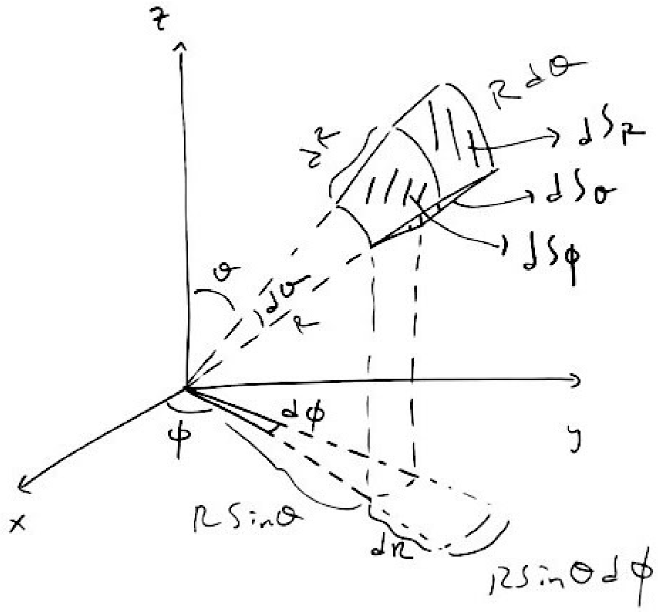
# Differentials:




Diff. Lengths : $dR$ , $Rd\theta$ , $R\sin\theta d\phi$

Diff. Areas:

$$dS_R = R^2 \sin\theta \, d\theta \, d\phi$$
$$dS_\theta = R\sin\theta \, dR \, d\phi$$
$$dS_\phi = R \, dR \, d\theta$$

Diff Length vector:

$$\bar{dl} = \hat{a}_R\, dR + \hat{a}_\theta\, R\, d\theta + \hat{a}_\phi\, R \sin\theta\, d\phi$$

Diff. area vectors:

$$d\bar{S}_R = \hat{a}_R\, dS_R$$
$$d\bar{S}_\theta = \hat{a}_\theta\, dS_\theta$$
$$d\bar{S}_\phi = \hat{a}_\phi\, dS_\phi$$

(Directions are normal to the surfaces)

Given a point $P(x, y, z)$ and $P(R, \theta, \phi)$ are related as

$$R^2 = x^2 + y^2 + z^2, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}$$

and

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$$

Vector Transformations:

Given a vector

$$\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z \quad \text{and}$$

$$\bar{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi \quad \text{are related as}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

and

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

# Vector Calculus:

## Vector Field:

Let D be a set in $\mathbb{R}^2$, A "vector field on $\mathbb{R}^2$" is a function F that assigns to each point $(x,y)$ in D a vector $F(x,y)$

Ex:

$$\bar{F}(x,y) = P(x,y)\hat{a}_x + Q(x,y)\hat{a}_y$$

where

$P(x,y)$ and $Q(x,y)$ are functions of x and y.

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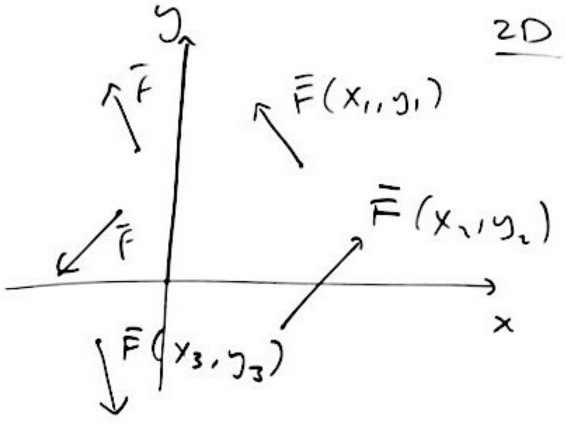



The definition can be extended to 3D.

The electric field and magnetic field are vector fields.

Integrals = Summation.

Line Integral

Surface Integrals.

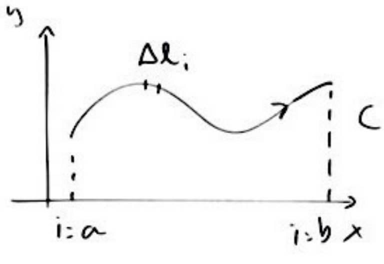
1-) Line Integrals (Curve Integrals)

Line Integral of Scalar Functions.

Line Integral of Vector Fields

☆

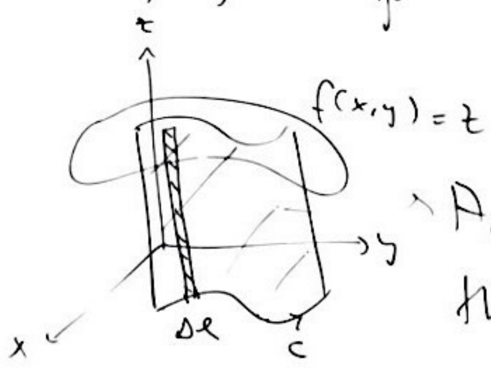
Line Integral of Scalar Functions:




The definition is

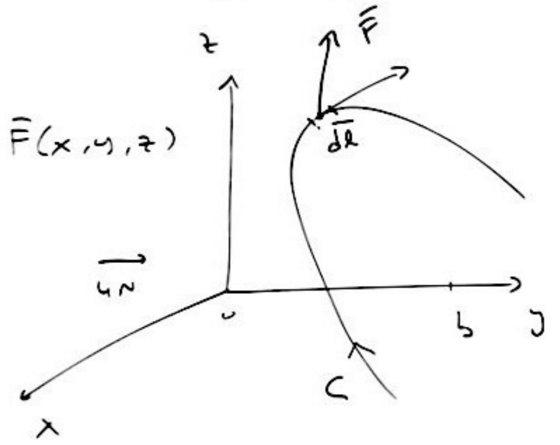
$$\lim_{\Delta l \to 0} \sum_{i=a}^{b} \underbrace{f(x_i, y_i)} \Delta l_i = \int_a^b f(x,y)\, dl$$

$f(x,y)$: Scalar function.




Area of the stripe is the line integral of scalar function b-) Line Integral of Vector Fields:

Curve.




The definition:

$$W = \int_C \bar{F}(x,y,z) \cdot \overline{dl}$$

$$= \int_0^b 4\hat{a}_y \cdot \hat{a}_y dy = \underline{4b} = \underline{F\, d}$$

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$d\bar{l}$ in this definition is the differential length vector and will be used as in rectangular, cylindrical or spherical coord. depending on the geometry of the problem.

The limits of the integral will be accordingly to the geometry.

Ex.

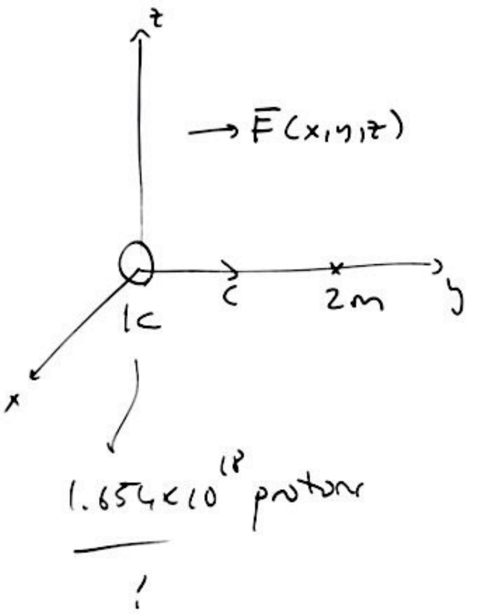
Given the force field $\bar{F}(x,y,z) = 4\,\hat{a}_y$ (N)

Determine the work done by

by the force field in moving a 1C of charge along

the y axis from y=0 to y=2m.

Ans:



Work $= \int_C \bar{F} \cdot \bar{dl}$

where $\bar{F} = \bar{F}(x,y,z) = \hat{a}_y 4$ (N)

and $d\bar{l} = ?$

Use the rectangular coord.

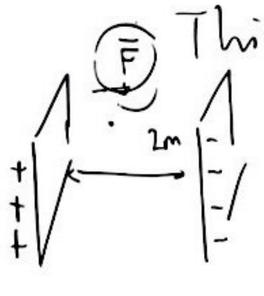
Then, $\bar{dl} = \hat{a}_y \, dy$

Thus,

$$W = \int_C (\hat{a}_y 4) \cdot (\hat{a}_y \, dy) = \int_0^2 4 \, dy$$

$$= 4 \cdot 2 = 8 \text{ joules}$$

This is the work per 1C of charge.




$$V = \frac{W}{C} \quad \text{(Volts)}$$

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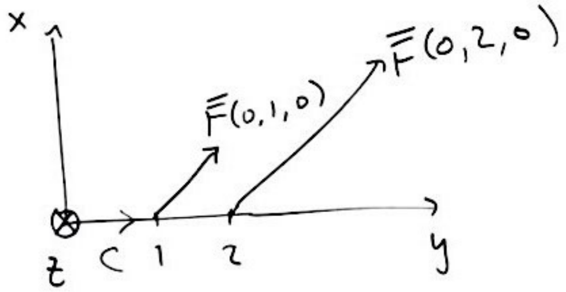
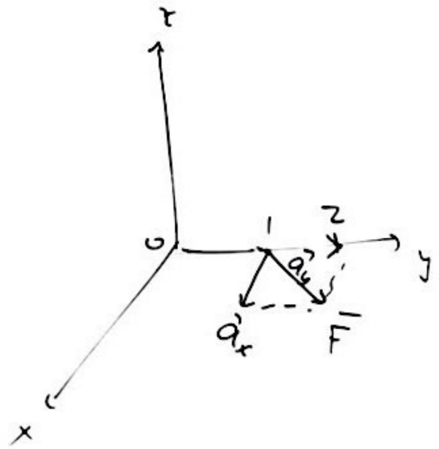
Given the force field

$$\bar{F}(x,y,z) = y\,\hat{a}_x + y\,\hat{a}_y$$

Determine the work by the force field in moving a 1C of charge along the y-axis from $y=0$ to $y=2$.

<u>Ans:</u>



$$W = \int_C \bar{F} \cdot \bar{dl}$$

$$\bar{F} = y\,\hat{a}_x + y\,\hat{a}_y$$

$$\bar{dl} = \hat{a}_y\, dy$$

Thus,

$$W = \int (y\,\hat{a}_x + y\,\hat{a}_y) \cdot (\hat{a}_y\, dy)$$

$$= \int_0^2 y\, dy = \frac{1}{2} y^2 \Big|_0^2 = \frac{1}{2} \cdot 4 = \underline{2 \text{ Volt}}.$$

image (21).pdf

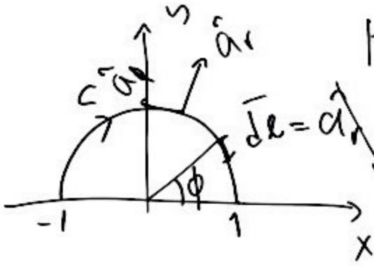
Ex:

Given the force field

$$\bar{F}(x,y) = x\,\hat{a}_x + y\,\hat{a}_y$$

Determine the work done by this force field in moving a 1C of charge along the contour defined as:

C:




Half unit circle.

$$\bar{dl} = \hat{a}_r \underbrace{dr}_{0} + \underline{\hat{a}_\phi\, r\, d\phi} + \hat{a}_z \underbrace{dz}_{0}$$

$$\bar{dl} = \hat{a}_\phi\, r\, d\phi = \hat{a}_\phi\, d\phi$$

Ans:

Use the cylindrical or spherical coord.

In cylindrical coord.

$\bar{dl} = \hat{a}_\phi \, d\phi$

Thus,

$$W = \int_c \bar{F} \cdot \bar{dl} = \int (x\hat{a}_x + y\hat{a}_y) \cdot (\hat{a}_\phi \, d\phi)$$

Transform $\bar{F} = (x\hat{a}_x + y\hat{a}_y)$ vector into cylindrical form.

$$\bar{F} = \underbrace{x}_{F_x}\hat{a}_x + \underbrace{y}_{F_y}\hat{a}_y$$

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Given that

$$\begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

The $\bar{F} = \hat{a}_r F_r + \hat{a}_\phi F_\phi + \hat{a}_z F_z$

we only need $F_\phi$ since $d\bar{l} = \hat{a}_\phi d\phi$

Then

$$F_\phi = -\sin\phi \, F_x + \cos\phi \, F_y$$

$$= -\sin^2\phi + \cos\phi \sin\phi$$

Then,

$$W = \int_0^\pi F_\phi \, d\phi = \int_0^\pi \left( -\sin^2\phi + \frac{2\cos\phi \sin\phi}{2} \right) d\phi$$

$$=$$

image (24).pdf

$$\int_0^{\pi} 1-\frac{\cos 2\phi+1}{2}+\frac{\sin 2\phi}{2}\,d\phi$$

$$\pi-\int_0^{\pi}\frac{\cos 2\phi}{2}\,d\phi+\int_0^{\pi}\frac{1}{2}\,d\phi+\frac{1}{2}\int_0^{\pi}\sin 2\phi\,d\phi$$

$$\pi-\frac{\cancel{\sin 2\phi}^{\,0}}{\cancel{4}}\Big|_0^{\pi}+\frac{\phi}{2}\Big|_0^{\pi}-\frac{1}{2}\,\frac{\cos 2\phi}{2}\Big|_0^{\pi}$$

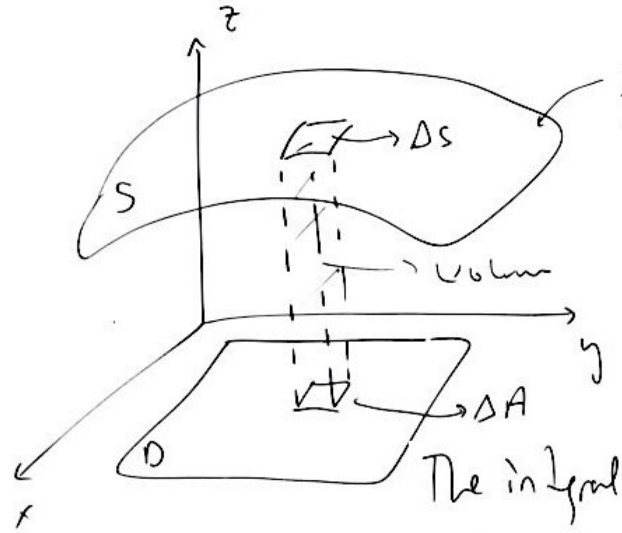
$$\pi+\frac{\pi}{2}=\frac{3\pi}{2}$$

## 2.) Surface Integrals

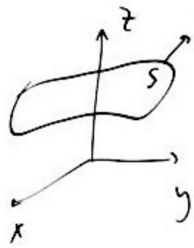
### a.) Surface Integrals of Scalar Functions:

By definition

$$\lim_{|P| \to 0} \sum_i \sum_j f(P_{ij}) \Delta S_{ij} = \iint_S f(x,y)\, dS$$




$f(x,y,z)=0$ or $f(x,y)=z$ defines a surface.

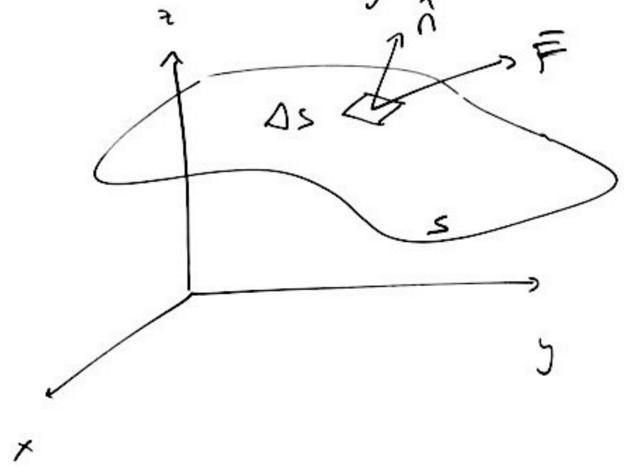



$f(x,y)=z$

$$\underbrace{f(x,y)-z=0}_{f(x,y,z)=0}$$

The integral gives the volume enclosed by the surface S and domain D.

b) Surface Integral of vector fields:

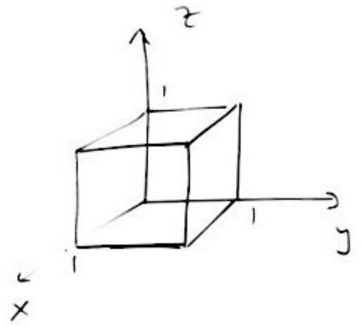



By definition

$$\text{Flux} = \iint_S \bar{F} \cdot d\bar{s} = \iint_S \bar{F} \cdot \hat{n}\, ds$$

Ex

Evaluate the flux on a cube




Given that $\bar{F} = \hat{a}_x x^2 + \hat{a}_y xy + \hat{a}_z yz$

Ans:

$$\iint_S \bar{F} \cdot d\bar{s} = \iint_{S_1} \bar{F} \cdot d\bar{s}_1 + \iint_{S_2} \bar{F} \cdot d\bar{s}_2 + \dots + \iint_{S_6} \bar{F} \cdot ds_6$$

Front face: $x = 1$, $d\bar{s}_1 = \hat{a}_x dy dz$

$$\iint \bar{F} \cdot \underbrace{d\bar{s}_1}_{\hat{a}_x dy dz}$$

(surface vector diff. element)

Then

$$\int_0^1\int_0^1 \underset{\substack{\downarrow \\ 1}}{x^2}\, dy\,dz = \int_0^1\int_0^1 dy\,dz = 1$$

Back face: $x = 0$, $d\bar{s}_2 = -\hat{a}_x\, dy\, dz$

$$\int_0^1\int_0^1 \bar{F} \cdot d\bar{s} = 0$$

Left face: $y = 0$, $d\bar{s}_3 = -\hat{a}_y\, dx\, dz$

$$\int_0^1\int_0^1 \bar{F} \cdot d\bar{s}_3 = 0$$

Right face: $y = 1$, $d\bar{s}_4 = \hat{a}_y\, dx\, dz$

$$\int_0^1\int_0^1 x\underset{\substack{\downarrow \\ 1}}{y}\, dx\, dz = \int_0^1 \frac{1}{2} x^2 \Big|_0^1 dz$$

$$= \frac{1}{2}\int_0^1 dz = \frac{1}{2}$$

Top face: $z = 1$ , $\bar{ds}_5 = \hat{a}_z \, dx\, dy$

$$\int_0^1 \int_0^1 y\underset{\substack{| \\ 1}}{z} \, dx\, dy = \frac{1}{2}$$

Bottom face: $z = 0$, $\bar{ds}_6 = -\hat{a}_z \, dx\, dy$

$$\int_0^1 \int_0^1 \bar{F} \cdot \bar{ds}_6 = 0$$

Then, the total flux is

$$\text{flux} = 1 + 0 + 0 + \frac{1}{2} + \frac{1}{2} + 0 = 2$$

image (30).pdf

Ex:

Evaluate the flux $\iint_S \bar{F} \cdot d\bar{s}$

where

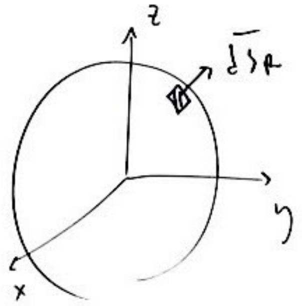
$$\bar{F} = R^2 \hat{a}_R$$

and $S$ is the region $x^2 + y^2 + z^2 = 1$ (sphere)

Ans:

$$\text{Flux} = \iint_S \bar{F} \cdot d\bar{s}$$

Use spherical coord. $\Rightarrow d\bar{s} = \hat{a}_R R^2 \sin\theta \, d\theta \, d\phi$

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$$\text{Flux} = \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= -\int_0^{2\pi} \underbrace{\cos\theta \Big|_0^{\pi}}_{\cos\pi - \cos 0 \;=\; -1 - 1} d\phi = 2\int_0^{2\pi} d\phi = 4\pi$$

In summary,

- Integrals
  - Line Integral
    - Scalar function
    - Vector field ☆
  - Surface Int
    - Scalar
    - vector field. ☆

The reason we are interested in the vector fields is that the
and magnetic fields (force fields) are <u>vector fields</u>.

Gradient Vector:

Define $\bar{\nabla}$ = Grad = Del

such that

$$\bar{\nabla} \cdot \underbrace{f(x,y)}_{\substack{\text{scalar} \\ \text{function}}} = \langle f_x(x,y), f_y(x,y) \rangle$$

is called the "Gradient of f".

Intuitively, it gives the rate of change vector at $x = x_0$, $y = y_0$.

For $f = f(x,y,z)$

$$\bar{\nabla} \cdot f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

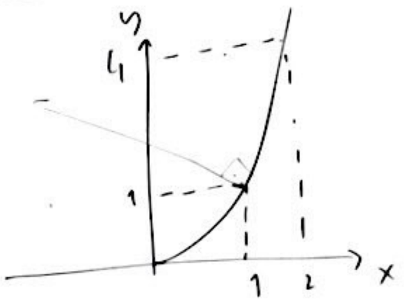
Note that Gradient vector is always perpendicular to the given point.

Ex:

$f(x,y) = y - x^2 = 0$

Find $\bar{\nabla} \cdot f \Big|_{(1,1)} = ?$

Ans:



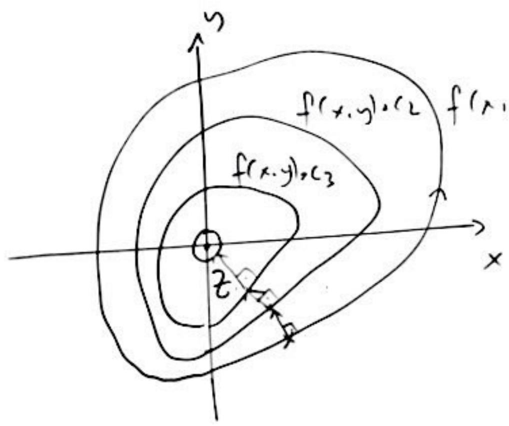

For example
at $x = 1$
$\bar{\nabla} f(1,1) = \langle -2, 1 \rangle$

$\bar{\nabla} f = \langle -2x, 1 \rangle$

image (34).pdf

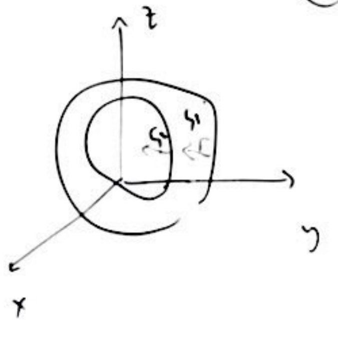
Suppose we have several functions

$f(x,y) = c_i$ = constant




In 3D,

$f(x,y,z) = c_i$ ← for example a constant temperature

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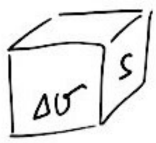
## Divergence of a Vector Field:

By definition, the divergence of a vector field is defined as

$$\operatorname{div} \bar{A} = \bar{\nabla} \cdot \bar{A} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{A} \cdot \bar{ds}}{\Delta v}$$

where



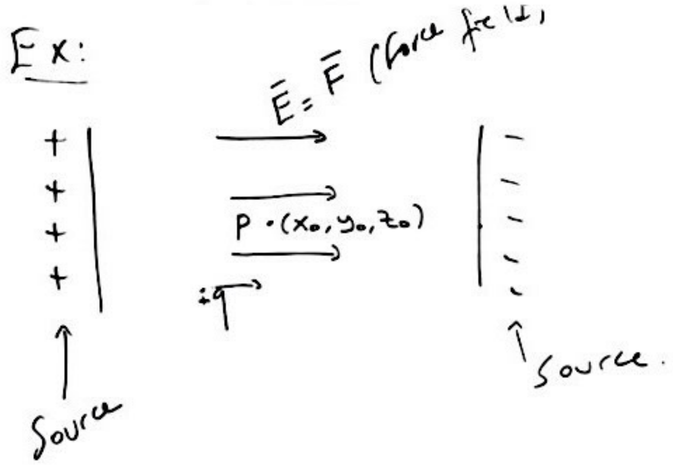

S : closed surface.

Δv = volume bounded by S.

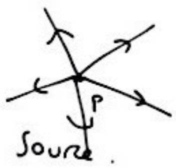
Intuitively, divergence gives us the total flux per unit volume as the unit volume becomes smaller and smaller (point)

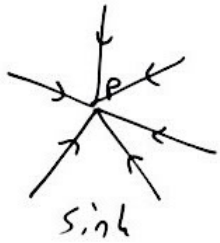
image (36).pdf

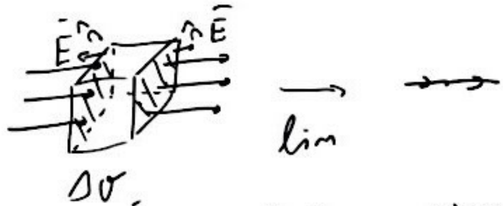
Ex:



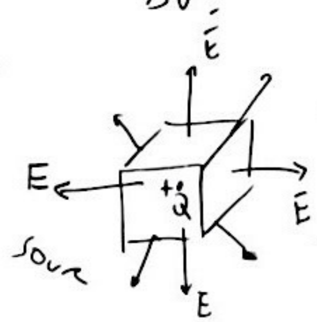

$$\operatorname{div} \bar{E} \Big|_{P(x_0, y_0, z_0)} = \bar{\nabla} \cdot \bar{E}(x_0, y_0, z_0)$$




$\bar{\nabla} \cdot \bar{E} = 0$ since the total flux = 0.




$\Rightarrow \bar{\nabla} \cdot \bar{E} \Big|_P \neq 0$

Thus, if the divergence is nonzero, this implies the existence of a source or a sink at the point P. If the divergence is zero at P, this means that there is no source at P.

Analatically, we will use the following formula for evaluating the flux.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

where

$$\bar{A} = \langle A_1, A_2, A_3 \rangle = \hat{a}_{u1} A_1 + \hat{a}_{u2} A_2 + \hat{a}_{u3} A_3$$

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where

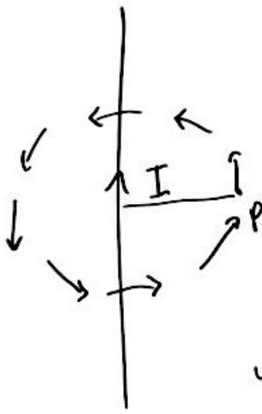
| | | (x, y, z) Cartesian | (r, $\phi$, z) Cylindrical | (R, $\theta$, $\phi$) Spherical |
|---|---|---|---|---|
| $\hat{a}_{u1}$ | = | $\hat{a}_x$ | $\hat{a}_r$ | $\hat{a}_R$ |
| $\hat{a}_{u2}$ | = | $\hat{a}_y$ | $\hat{a}_\phi$ | $\hat{a}_\theta$ |
| $\hat{a}_{u3}$ | = | $\hat{a}_z$ | $\hat{a}_z$ | $\hat{a}_\phi$ |
| $h_1$ | = | 1 | 1 | 1 |
| $h_2$ | = | 1 | $r$ | $R$ |
| $h_3$ | = | 1 | 1 | $R \sin\theta$ |

Ex:

The magnetic flux density $\bar{B}$ outside a very long current carrying wire is circumferential and is inversely proportional to the distance to the axis of the wire. Find $\bar{\nabla} \cdot \bar{B}$.

Ans:

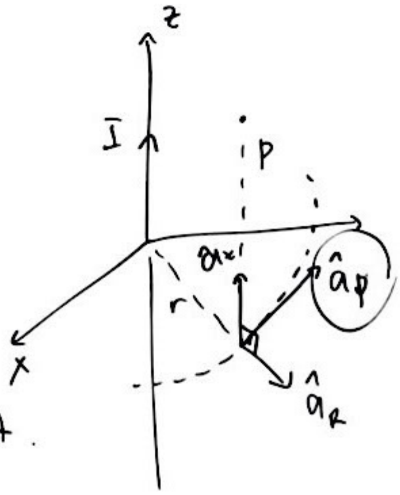
force field.





$$\bar{B} = \hat{a}_\phi B_\phi = \hat{a}_\phi \frac{k}{r}$$

where $k$ = constant.

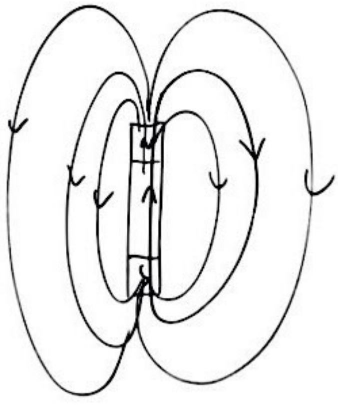


$$\bar{\nabla} \cdot \bar{B} = \frac{1}{r}\frac{\partial}{\partial r}(r \cancel{B_r}) + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial \cancel{B_z}}{\partial z}$$

$$\bar{\nabla} \cdot \bar{B} = \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} = \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{k}{r}\right) = 0 .$$

What does this result mean?

- No source outside
- Vector field $\bar{B}$ close upon itself.
  (solenoidal field)



Solenoidal.

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# Curl of a Vector Field:

By definition

$$\text{curl}\,\bar{A} = \bar{\nabla} \times \bar{A} = \lim_{\Delta s \to 0} \frac{\overbrace{\oint_C \bar{A} \cdot d\bar{l}}^{\text{Circulation}}}{\Delta s}$$

where

closed contour.

$$\Delta s \;\; C \xrightarrow[\lim]{} \text{point.}$$

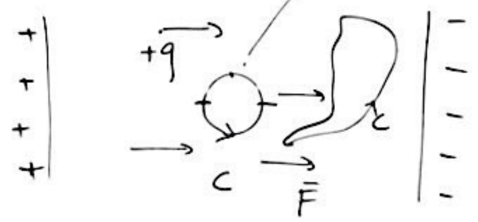
Analytically,

$$\bar{\nabla} \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{a}_{u_1} h_1 & \hat{a}_{u_2} h_2 & \hat{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

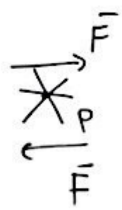
Intuitively, what does curl mean?

Ex:

Work = 0 (Conservative field)




$\bar{\nabla} \times \bar{F} = 0$ (Conservative field)




If the wheel rotates at P,

$\bar{\nabla} \times \bar{F} \neq 0$

If it doesn't rotate

$\bar{\nabla} \times \bar{F} = 0$ at P.
(Conservative field at P)

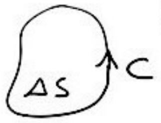
# Curl of a Vector Field:

By definition

$$\text{curl } \bar{A} = \bar{\nabla} \times \bar{A} = \lim_{\Delta s \to 0} \frac{\overbrace{\oint_C \bar{A} \cdot d\bar{l}}^{\text{Circulation}}}{\Delta s}$$

where

closed contour.



$\xrightarrow{\lim}$ point.

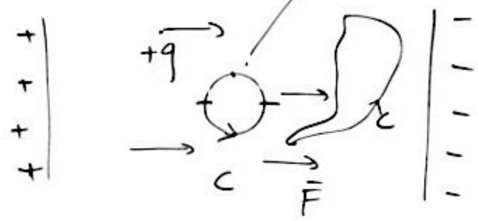
Analatically,

$$\bar{\nabla} \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{a}_{u_1} h_1 & \hat{a}_{u_2} h_2 & \hat{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

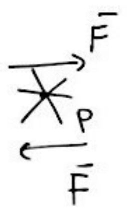
Intuitively, what does curl mean?

Ex:

Work = 0 (Conservative field)




$\bar{\nabla} \times \bar{F} = 0$ (Conservative field)




If the wheel rotates at P, $\bar{\nabla} \times \bar{F} \neq 0$

If it doesn't rotate $\bar{\nabla} \times \bar{F} = 0$ at P. (Conservative field at P)

**image (45).pdf**

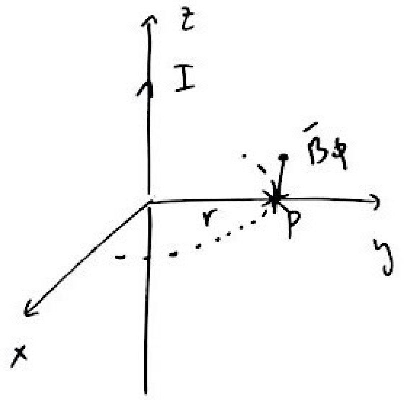
## Ex:

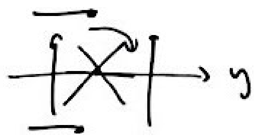
Find the curl of a vector field

$$\bar{B} = \hat{a}_\phi \frac{k}{r} = \hat{a}_r \cancel{B_r}^{\,0} + \hat{a}_\phi B_\phi + \hat{a}_z \cancel{B_z}^{\,0}$$

given in the previous question.

## Ans:



$$\bar{\nabla} \times \bar{B}\Big|_{\rho} = \frac{1}{r}\begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \underset{0}{\cancel{B_r}} & rB_\phi & \underset{0}{\cancel{B_z}} \end{vmatrix}$$

$$= \frac{1}{r}\left\{ \hat{a}_r\left[-\frac{\partial}{\partial z}(rB_\phi)\right] + r\hat{a}_\phi\left[0\right] + \hat{a}_z\left[\frac{\partial}{\partial r}(rB_\phi) - 0\right]\right\}$$

$$= \frac{1}{r}\left\{ \hat{a}_r\left[-\frac{\partial}{\partial z}\left(\cancel{r}\frac{k}{\cancel{r}}\right)\right] + r\hat{a}_\phi\left[0\right] + \hat{a}_z\left[\frac{\partial}{\partial r}\left(\cancel{r}\frac{k}{\cancel{r}}\right)\right]\right\}$$

$$= 0 \checkmark$$

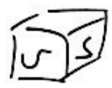
A curl-free vector field is called irrotational or conservative field.

HW#1: Why is this field is irrotational.

image (47).pdf

## Divergence Theorem:

$$\int_V \bar{\nabla} \cdot \bar{A} \cdot dv = \oint_S \bar{A} \cdot d\bar{s}$$



## Stoke's Theorem:

$$\int_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{s} = \oint_C \bar{A} \cdot d\bar{l}$$

Also, note that $\bar{\nabla} \cdot \bar{A}$ is scaler and $\bar{\nabla} \times \bar{A}$ is a vector quatities.

## Identities:

1-) $\bar{\nabla} \times (\bar{\nabla} V) = 0$

or

$\bar{E} = -\bar{\nabla} V$

The curl of a vector field $\bar{E}$ is curl-free. Then $\bar{E}$ can be written in terms of the gradient of a scaler (V).

2-) $\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0$

Divergence of the curl of any vector field $\bar{A}$ is zero.

$\bar{A}$ can not be conservative and rotational at the same time.

or vise versa.

image (49).pdf

# Helmholtz Theorem:

1-) Solenoidal and irrotational

$$\bar{\nabla}\cdot\bar{F}=0 \quad \text{and} \quad \bar{\nabla}\times\bar{F}=0$$

Ex. Static Electric field in charge-free region

2-) Solenoidal but not irrotational

$$\bar{\nabla}\cdot\bar{F}=0 \quad \text{and} \quad \bar{\nabla}\times\bar{F}\neq 0$$

Ex: Static magnetic field in current carrying conductor

3-) Irrotational but not solenoidal

$$\bar{\nabla}\times\bar{F}=0 \quad \text{and} \quad \bar{\nabla}\cdot\bar{F}\neq 0$$

Ex: Static electric field in charged region

4-) Neither solenoidal nor irrotational

$$\bar{\nabla}\cdot\bar{F}\neq 0 \quad \text{and} \quad \bar{\nabla}\times\bar{F}\neq 0$$

Ex: Electric field in charged region with time-varying magnetic field.

Helmholtz's Theorem: A vector field is determined

if both its divergence and curl are specified everywhere.

Ex:

Given a vector function

$$\bar{F} = \hat{a}_x(3y - c_1 z) + \hat{a}_y(c_2 x - 2z) - \hat{a}_z(c_3 y + z)$$

Determine the constants $c_1$, $c_2$, and $c_3$ if $\bar{F}$ is irrotational.

Ans:

$\bar{\nabla} \times \bar{F} = 0$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y - c_1 z & c_2 x - 2z & -(c_3 y + z) \end{vmatrix}$$

$$= \hat{a}_x(-c_3 + 2) - \hat{a}_y c_1 + \hat{a}_z(c_2 - 3) = 0$$

$\Rightarrow$ $c_1 = 0$, $c_2 = 3$, $c_3 = 2$.

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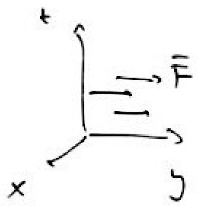
Until now, we reviewed;

- Vectors → summation, multiplication

  dot ↙ ↘ cross.

- Rectangular, cylindrical, spherical coord.

- Vector Fields.




← Every point, the function F assigns a vector.

- Integrals = Summation

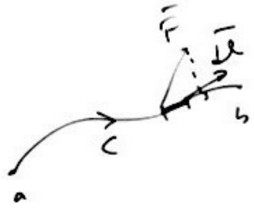
Line

Work = $\int_C F \cdot d\bar{l}$

↑ Vector field.

Surface.

Flux = $\int_S \bar{F} \cdot d\bar{s}$





- Divergence $\bar{\nabla}\cdot\bar{F}$ = Scalar
- Curl $\bar{\nabla}\times\bar{F}$ = Vector field.

point forms.

If $\bar{\nabla}\cdot\bar{F} = 0 \longrightarrow$ Source-free

$\bar{\nabla}\cdot\bar{F}(x_0, y_0, z_0)$

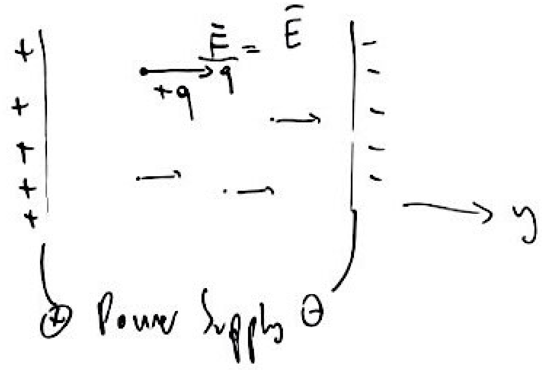
If $\bar{\nabla}\times\bar{F} = 0 \longrightarrow$ irrotational.

## Static Electric Field : (Chp. 3)

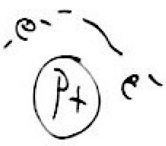
What is the Electric Field?

E-field = Force Field = Vector Field.

Ex:




$1C = 1.654 \times 10^{18}$




$\bar{E} = \frac{\bar{F}}{q}$ = Force field per charge. $\left(\frac{N}{C}\right)$

Ex:

If the $\bar{E} = \hat{a}_y 4 \ (\frac{N}{C})$

If we put 1C of charge, it would experience

$\bar{F} = \bar{E}.Q = \hat{a}_y 4$ Newton.

If we put 2C of charge,

$\bar{F} = \bar{E}.Q = \underline{8\hat{a}_y} \ (N)$

---

There are two equations obtained by measurement or lab experiments (postulates) for static electric field,

↳ ? The electric field does not change with time.

$\frac{d}{dt} = 0$

image (55).pdf

1-) $\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0}$ (Divergence depends on the source at $P(x_0, y_0, z_0)$).

and

2-) $\bar{\nabla} \times \bar{E} = 0$ (irrotational)

Point form

where

$\epsilon_0 = \text{constant} = \frac{1}{36\pi \times 10^9} = 8.854 \times 10^{-12}$

and

$\rho =$ volume charge density $\left(\frac{C}{m^3}\right)$

In solving problems, these two equations are difficult to use, thus we sum or integrate both sides, then we get

$$\int_V \bar{\nabla}\cdot\bar{E}\, dv = \int_V \frac{\rho}{\epsilon_0}\, dv$$

by divergence theorem

1.) $\boxed{\oint_S \bar{E}\cdot d\bar{s} = \frac{Q}{\epsilon_0} \quad \text{(Gauss law)}}$ (Integral form)

where S is a closed surface that bounds the volume V. The Q is the total charge inside S.

Similarly, if we integrate both sides of the second equation, we get

$$\int_S (\bar{\nabla} \times \bar{E}) \cdot d\bar{s} = 0$$

By Stokes' Theorem

or

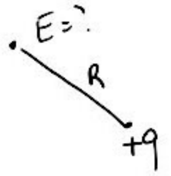
2-3 $$\boxed{\oint_C \bar{E} \cdot d\bar{l} = 0}$$ (Integral form)

where the contour C closes the surface S.


meaning that the static $\bar{E}$-field is conservative.

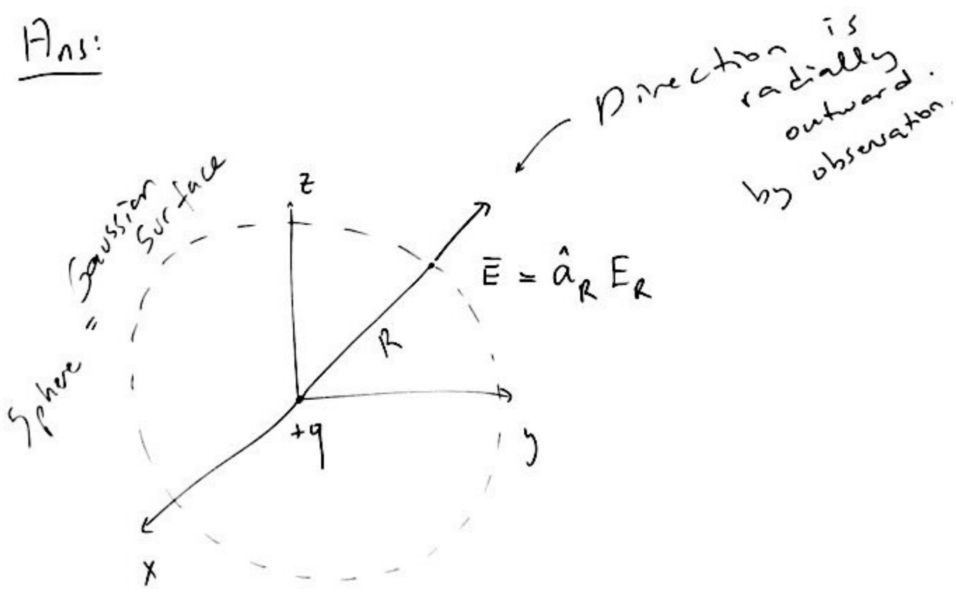
Ex:

Find the electric field created by a static charge $q$.




$q \neq q(t)$

Ans:

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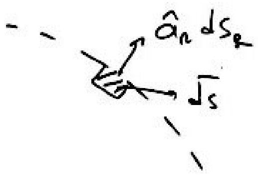
$$\oint_S \bar{E} \cdot d\bar{s} = \frac{Q}{\epsilon_0}$$

$\bar{E} \rightarrow \hat{a}_R E_R$

$d\bar{s} \rightarrow \hat{a}_R R^2 \sin\theta \, d\theta \, d\phi$




$$\int_0^{2\pi}\int_0^{\pi} (\hat{a}_R E_R) \cdot (\hat{a}_R R^2 \sin\theta \, d\theta \, d\phi) = \frac{q}{\epsilon_0}$$

$$\int_0^{2\pi}\int_0^{\pi} E_R R^2 \sin\theta \, d\theta \, d\phi = \frac{q}{\epsilon_0}$$

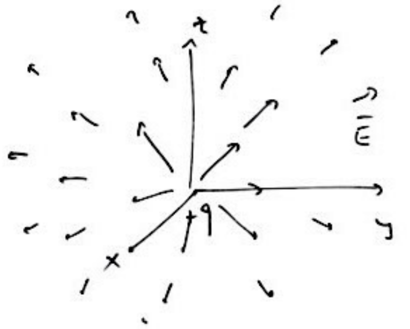
Since $E_R \neq E_R(\theta, \phi)$

Hence can be taken out of the integral

$$E_R \underbrace{\int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta \, d\theta \, d\phi}_{4\pi R^2} = \frac{q}{\epsilon_0}$$

Thus,

$$\bar{E} = \hat{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad \left(\frac{N}{C}\right)$$




From this result, we can calculate the force as

$$\bar{F} = \hat{a}_R \frac{q_t\, q}{4\pi\epsilon_0 R^2} \quad (N)$$

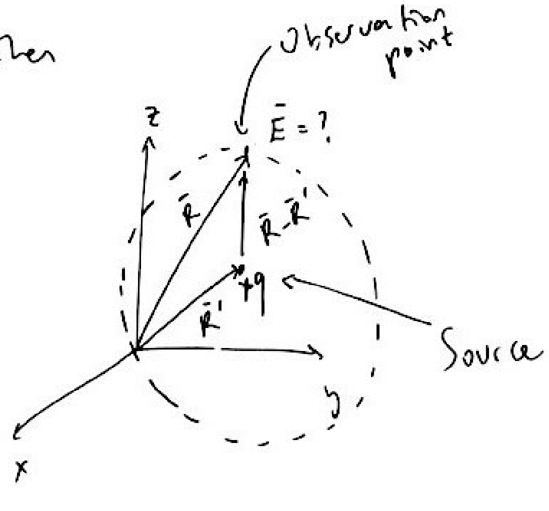
(Coulomb's law)

Note that the Gaussian surface must be selected such that

1-) the Electric field on the surface must have the same magnitude and direction everywhere on the surface.

2-) It must be a symmetrical surface.

If the source charge is not placed at the origin then




$$\bar{E} = \frac{q\,(\bar{R} - \bar{R}')}{4\pi\epsilon_0 |\bar{R} - \bar{R}'|^3}$$

Ex:

Determine the E-field at $P(-0.2, 0, -2.3)$ due to a point charge of $+5nC$ at $Q(0.2, 0.1, -2.5)$ in air.

Ans:

$$\bar{R} = \overrightarrow{OP} = -\hat{a}_x 0.2 - \hat{a}_z 2.3$$

$$\bar{R}' = \overrightarrow{OQ} = \hat{a}_x 0.2 + \hat{a}_y 0.1 - \hat{a}_z 2.5$$

$$\bar{R} - \bar{R}' = -\hat{a}_x 0.4 - \hat{a}_y 0.1 + \hat{a}_z 0.2$$

$$|\bar{R} - \bar{R}'| = \left[(-0.4)^2 + (-0.1)^2 + (0.2)^2\right]^{\frac{1}{2}} = 0.458 (m)$$

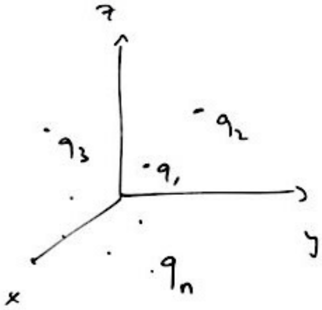
Then,

$$\bar{E} = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{q(\bar{R} - \bar{R}')}{|\bar{R} - \bar{R}'|^3} = 214.5\left(-\hat{a}_x 0.873 - \hat{a}_y 0.218 + \hat{a}_z 0.437\right) \left(\frac{N}{C}\right)$$

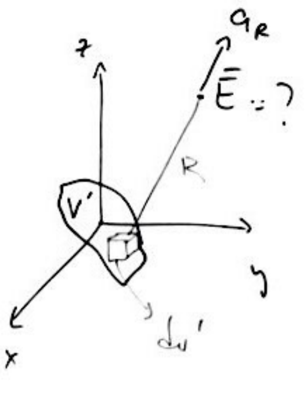
image (63).pdf

Electric Field due to System of Discrete charges:

$$\bar{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k(\bar{R}-\bar{R}'_k)}{|\bar{R}-\bar{R}'_k|^3} \quad \text{(for n discrete charge.)}$$




Electric Field due to Continous Distribution of Charge:

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Total charge inside $dv'$ is

$\underline{\rho\, dv'}$

The electric field due to $\rho\, dv'$ is

let's say

$$d\bar{E} = \hat{a}_R \frac{\rho\, dv'}{4\pi\epsilon_0 R^2}$$

The electric field due to $v'$

is the summation of $d\bar{E}$'s.

Then,

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{v'} \hat{a}_R \frac{\rho}{R^2}\, dv'}$$

Since $\hat{a}_R = \frac{\bar{R}}{R}$

$$\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{v'} \rho \frac{\bar{R}}{R^3}\, dv'$$

**image (65).pdf**

If the source exists on a surface,

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \hat{a}_R \frac{\rho_s}{R^2}\, ds'}$$ , where $\rho_s = \left(\frac{C}{m^2}\right)$ = Surface charge density.

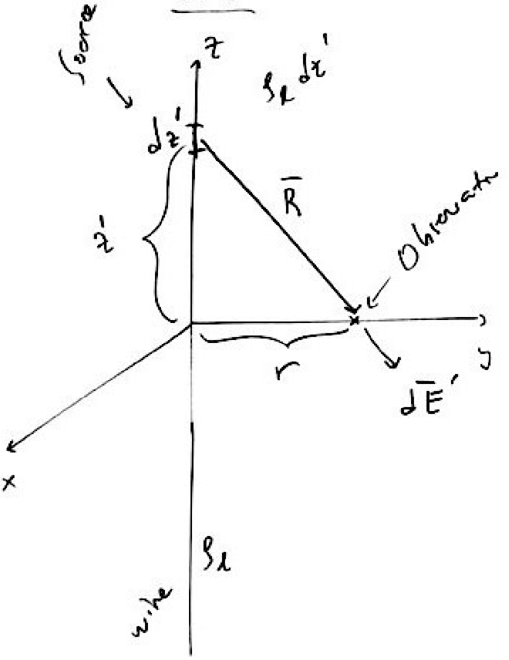
If the source is on a line (not necessarily straight)

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \hat{a}_R \frac{\rho_\ell}{R^2}\, d\ell'}$$ , $\rho_\ell$ = Line charge density $\left(\frac{C}{m}\right)$

Ex:

Determine the $\bar{E}$-field of an infinitely long, straight, line charge of uniform density $\rho_\ell$ in air.

## Ans:




$$\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{L'} \rho_\ell \frac{\bar{R}}{R^3} \, d\ell'$$

Thus, we need $\bar{R}$.

$$\bar{R} = \hat{a}_r r - \hat{a}_z z'$$

Then, $$d\bar{E} = \frac{\rho_\ell \, dz'}{4\pi\epsilon_0} \frac{(\hat{a}_r r - \hat{a}_z z')}{(r^2 + z'^2)^{3/2}}$$

$d\bar{E} = d\bar{E}_r + \cancel{d\bar{E}_z}$ drops since the contributions from $-z$ and $+z$ cancels out.

$$dE_r = \frac{\rho_\ell r \, dz'}{4\pi\epsilon_0 (r^2 + z'^2)^{3/2}} \Rightarrow \bar{E} = \hat{a}_r \frac{\rho_\ell r}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{(r^2 + z'^2)^{3/2}}$$

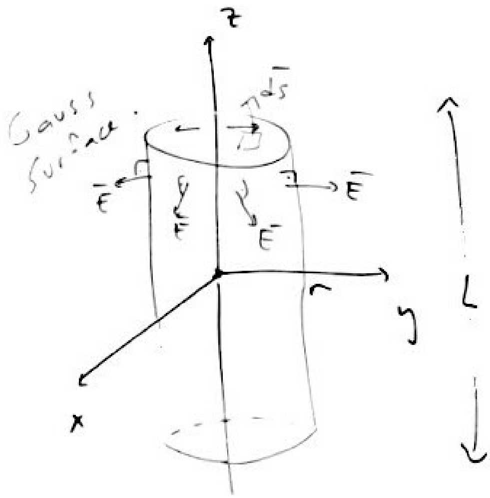
**image (67).pdf**

$$\bar{E} = \hat{a}_r \frac{\rho_L}{2\pi\epsilon_0 r} \quad \left(\frac{\sim}{c}\right)$$

If $r \ll$ Wire length then this result is a good approximation to the real problem.

The same problem would be solved by Gauss' law

Ex:




$$\oint_s \bar{E} \cdot \bar{ds} = \frac{Q}{\epsilon_0}$$

Gauss Surface.

1.) Symmetry w.r.t. coord. variables.

2.) On the surface $\bar{E}$-field must be constant.

$$\iint \bar{E} \cdot \bar{ds} = \frac{Q}{\epsilon_0}$$

where

$$\bar{E} = \hat{a}_r E_r \quad , \quad \bar{ds}_r = \hat{a}_r \, r \, d\phi \, dz$$

and

$$Q = \rho_\ell L \quad (C)$$

Then

$$\int_0^L \int_0^{2\pi} \hat{a}_r E_r \cdot \hat{a}_r \, r \, d\phi \, dz = \frac{\rho_\ell L}{\epsilon_0}$$

$$\int_0^L \int_0^{2\pi} E_r \, r \, d\phi \, dz = \frac{\rho_\ell L}{\epsilon_0}$$

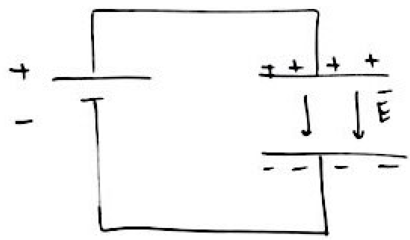
$$E_r r \int_0^L \int_0^{2\pi} d\phi \, dz = \frac{\rho_\ell L}{\epsilon_0}$$

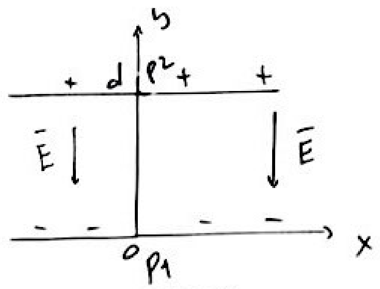
$$E_r r \cancel{L} 2\pi = \frac{\rho_\ell \cancel{L}}{\epsilon_0} \Rightarrow \boxed{E_r = \frac{\rho_\ell}{2\pi\epsilon_0 r} \quad \left(\frac{N}{C}\right)}$$

**image (69).pdf**

This result is the same as before, the Gauss Law is easier, but it needs a Gaussian Surface which is not always easy to obtain.

<u>Electric Potential:</u> (Voltage)




By definition
$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \bar{E} \cdot d\bar{l}}$$
(Potential difference)

$$V = \frac{\int_{1}^{2} \bar{F} \cdot d\bar{l}}{q} = \frac{W}{q} \quad \left(\frac{\text{joule}}{\text{colomb}} = V\right)$$

Hence, the volt (electric potential) is the potential energy per charge.

<u>Ex:</u>

$$\bar{E} = -\hat{a}_y E_y$$

and

$$d\bar{l} = \hat{a}_y \, dy .$$

Then

$$V = V_2 - V_1 = -\int_0^d (-\hat{a}_y E_y) \cdot (\hat{a}_y \, dy)$$

$$= \int_0^d E_y \, dy = d E_y = E \cdot d$$

Uniform $\bar{E}$ field over the region.

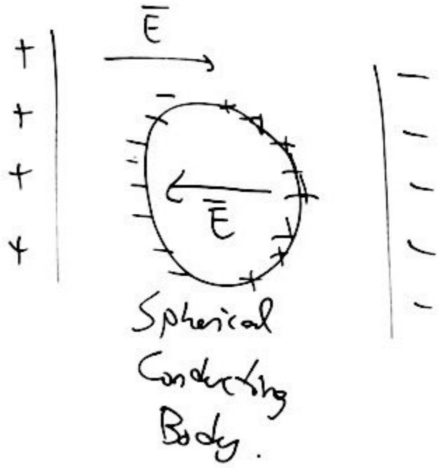
image (71).pdf

Take the derivative of both sides of

$$V = -\int_{P_1}^{P_2} \bar{E} \cdot \bar{dl}$$

$$\boxed{\bar{E} = -\bar{\nabla} V}$$

Conductors in <u>Static</u> Electric Field:

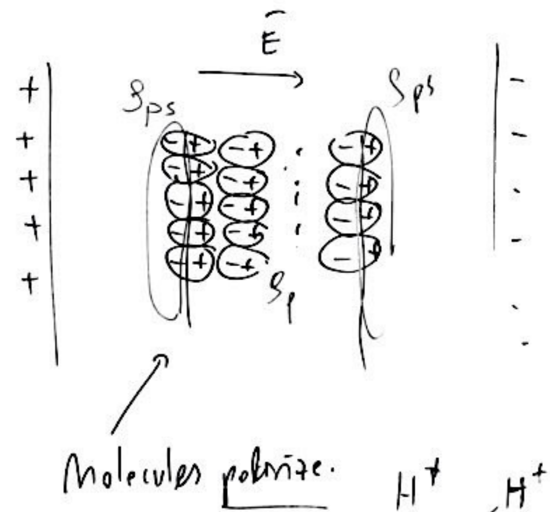



Inside the conductors:

$$\boxed{\bar{E} = 0}, \quad \boxed{\rho = 0}$$

# Dielectrics in Static Electric Field:

- Dielectrics do not contain free-charges.
- Conductor (metals) do contain many free-charges.




$H^+$ $H^+$ $O_{-2}$

Define dipole moment:


$\bar{p}$ = dipole moment

$$\boxed{\bar{p} = \bar{d} \cdot q}$$

Define the polarization vector:

$$\bar{P} = \lim_{\Delta v \to 0} \frac{\sum_{k}^{n\Delta v} \bar{p}_k}{\Delta v}$$ = volume density of dipole moments.

⋮ After some manipulations;

Polarization surface charge density — $\rho_{ps} = \bar{P} \cdot \hat{a}_n$ , $\rho_p = -\bar{\nabla} \cdot \bar{P}$ → Polarization volume charge density

The fundamental eqn. for electrostatics,

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0} = \frac{(\overset{\text{free charge density}}{\rho} + \overset{\text{polarization charge density.}}{\rho_p})}{\epsilon_0}$$

$$\bar{\nabla}\cdot\epsilon_0\bar{E} = \rho - \bar{\nabla}\cdot\bar{P}$$

$$\bar{\nabla}\cdot\underbrace{(\epsilon_0\bar{E} + \bar{P})} = \rho$$

$\bar{D} = \epsilon_0\bar{E} + \bar{P}$ is called the displacement vector or electric flux density.

and

$\boxed{\bar{\nabla}\cdot\bar{D} = \rho}$ is more general form.

or

$\boxed{\oint_S \bar{D}\cdot\bar{ds} = Q}$ (General Gauss law)
(Includes the effect of the polarization)

$$\bar{P} = \epsilon_0 \chi_e \bar{E}$$

$\chi_e$ : Electric Susceptibility

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} = \epsilon_0 \bar{E} + \epsilon_0 \chi_e \bar{E}$$

$$= \epsilon_0 \underbrace{(1 + \chi_e)}_{\epsilon_r} \bar{E}$$

$$= \underbrace{\epsilon_0 \epsilon_r}_{\epsilon} \bar{E}$$

$$= \epsilon \bar{E}$$

where $\epsilon$ = Permittivity, and $\epsilon_r$ = Relative permittivity.

Relative Permittivies of some materials:

| | $\epsilon_r$ | | $\epsilon_r$ |
|---|---|---|---|
| Free space | 1 | | |
| Air | $1.00059 \approx 1$ | water (distilled) | 80 |
| Glass | 4-10 | Seawater | 72 |
| Paper | 2-4 | | |
| Rubber | 2.3-4 | | |

image (76).pdf

In summary,

Electric Field ?

Force Field acting on charges.

Vector field

$$\bar{E}(x,y,z) = \hat{a}_x \underbrace{E_x(x,y,z)}_{\substack{\text{functions} \\ \text{of } x, y \text{ and } z.}} + \hat{a}_y \underbrace{E_y(x,y,z)} + \hat{a}_z \underbrace{E_z(x,y,z)}$$

$$\bar{E}(x,y,z) = \frac{\bar{F}(x,y,z)}{q} \quad (\text{force field})$$

---

## Static Electric Field:

No time variation ($\frac{\partial}{\partial t} = 0$)

Two equations (for $\bar{E}$ computation)

1-) $\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0}$ $\longrightarrow$ $\oint_S \bar{E} \cdot \bar{ds} = \frac{Q}{\epsilon_0}$

2-) $\bar{\nabla} \times \bar{E} = 0$ $\longrightarrow$ $\oint_C \bar{E} \cdot \bar{dl} = 0$

- Usually we use Gauss' law to solve problems.
- We must define a Gaussian surface.
- If we can't define such surface, then we can use the integral techniques

$\Downarrow$

$\Downarrow$

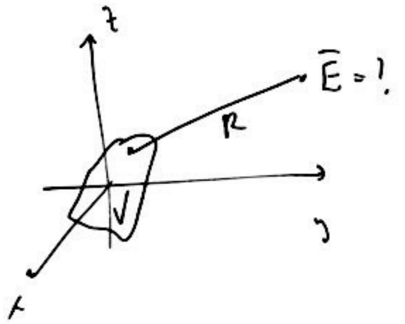
$$\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{v'} \hat{a}_R \frac{\rho}{R^2} \, dv'$$

or

$$\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{s'} \hat{a}_R \frac{\rho_s}{R^2} \, ds'$$

or

$$\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{c'} \hat{a}_R \frac{\rho_l}{R^2} \, dl'$$

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$$\text{Voltage} = \frac{W}{q} \quad \left( \frac{\text{Joule}}{\text{Coulomb}} = \text{Volt} \right)$$

Work.

$$V = -\int_{P_1}^{P_2} \bar{E} \cdot \bar{dl} = \frac{1}{q} \left( -\int_{P_1}^{P_2} \bar{F} \cdot \bar{dl} \right)$$

$$\bar{E} \rightarrow \frac{\bar{F}}{q}$$

$\Sigma$ force . distance

or

$$\bar{E} = -\bar{\nabla} V$$

---

Conductors in Electric field:

↳ Many free-electrons.

Inside the conductors

$$\bar{E} = 0$$

Dielectrics in Electric Field:

↳ Not many free-electrons.
(insulators, ... etc.)

When dielectrics are put inside static electric field, they polarize.

Due to the polarization, the electric field inside the dielectric is decreased.

The measure of polarization is the permittivity constant. ($\epsilon$)

$$\underbrace{\epsilon_r}_{\text{Relative permittivity}} \quad \epsilon = \epsilon_r \epsilon_0 \quad \epsilon_0 \text{: constant}$$

Free Space $\rightarrow \epsilon_r = 1$

Air $\rightarrow \epsilon_r \approx 1$

$\vdots$

High Polarized $\rightarrow$ Water $\rightarrow \epsilon_r = 70$

**image (81).pdf**

In solving the $\bar{E}$-field inside dielectrics we use

$$\bar{D} = \epsilon \bar{E}$$

Displacement vector

$$\epsilon_0 \bar{E} \rightarrow \epsilon_r \epsilon_0 \bar{E} = \epsilon \bar{E} = \bar{D}$$

and

$$\oint_S \bar{E} \cdot d\bar{s} = \frac{Q}{\epsilon_r \epsilon_0} = \frac{Q}{\epsilon}$$

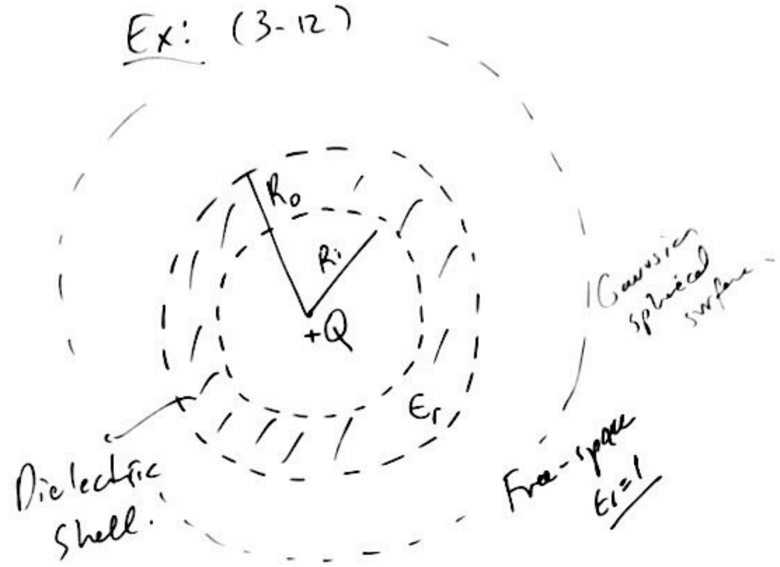
$$\oint_S \epsilon \bar{E} \cdot d\bar{s} = Q$$

$$= \boxed{\oint_S \bar{D} \cdot d\bar{s} = Q}$$ (General Gauss' law)

and also,

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P}$$

Ex: (3-12)




Determine $\bar{E}$, $\bar{D}$ and $\bar{P}$ everywhere.

Ans:

$R > R_o$:

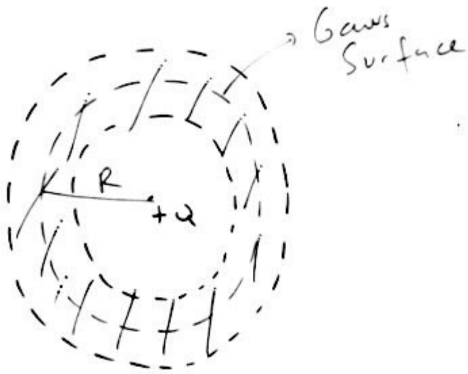
$$E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2} \left(\frac{V}{m}\right) \text{ as before.}$$

$$D_{R1} = \epsilon_0 \bar{E}_{R1} = \frac{Q}{4\pi R^2} \quad , \quad \bar{P}_{R1} = 0$$

image (83).pdf

In the region

$R_i < R < R_o$:




$$\oint_S \bar{D} \cdot d\bar{s} = Q$$

$$\int_0^{2\pi}\int_0^{\pi} \epsilon_r \epsilon_0 \underbrace{\bar{E}}_{\hat{a}_R E_{R2}} \cdot \hat{a}_R R \sin\theta \, d\theta \, d\phi = Q$$

$$\epsilon_r \epsilon_0 E_{R2} \underbrace{\int_0^{2\pi}\int_0^{\pi} R \sin\theta \, d\theta \, d\phi}_{4\pi R^2 \text{ (Area of the Gauss sphere)}} = Q \rightarrow \bar{E}_{R2} = \frac{Q}{4\pi \epsilon_0 \epsilon_r R^2} \left(\frac{V}{m}\right)$$

$$D_{R2} = \frac{Q}{4\pi R^2}$$

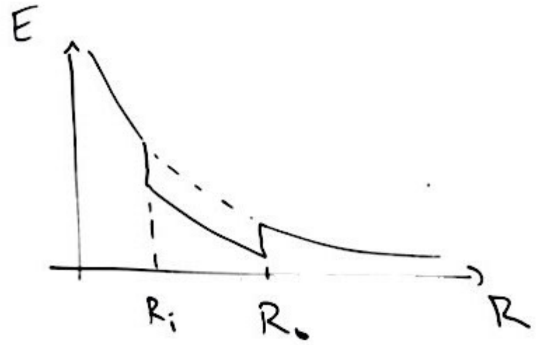
$$P_{R2} = \left(1 - \frac{1}{\epsilon_r}\right) \frac{Q}{4\pi R^2}$$

For $R < R_i$:

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$D_{R3} = \frac{Q}{4\pi R^2}$$

$$P_{R3} = 0 .$$

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